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# The Theory of Determinants: Some Special Forms and Applications

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THE THEORY OF DETERMINANTS:

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# CHAPTER I

## INTRODUCTION

The theory of determinants is relatively a new branch of mathematics; it was not firmly established until a brief 200 years ago. As in many other branches of science, the theory of determinants was rediscovered many times before it was permanently established. It was not until the middle of the 18th century that Cramer, one of the independent discoverers, of the fundamental idea, was fortunate enough to attract attention to the theory of determinants.

In the study and application of mathematics one often has occasion to use determinants, but frequently it is necessary to use many references to find an adequate treatment of the theorem in question. Much of the literature is written on the assumption that its reader has a good background in the subject. Then other authors just give the fundamental properties of determinants without proof of their statements. These latter authors are not interested in the mathematical theory of determinants, but are interested only in using them as a tool. Thus many terms are used without first giving a definition of them and theorems are used without proof; as a consequence it is necessary for the beginner to look elsewhere for these definitions and proofs.

In this thesis no attempt is made to cover the entire field of the theory of determinants, but an attempt is made to summarize the material in certain phases of the theory of determinants. This monograph is divided into the following parts:- (1) A brief history of the early developments in the theory of determinants. (2) The statement and proof of the general properties of determinants, and the definition of the common terms. (3) A brief treatment of the most common special forms of determinants. (4) A few of the many applications of determinants and an indication of others in various phases of mathematics and other sciences.

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## CHAPTER II

## A BRIEF HISTORY OF DETERMINANTS

As in many other fields of science it is difficult to determine who was the first person to consider the theory of determinants. The honor has been attributed to Leibnitz (1693); however, Cajori[1] and Smith[2] each state that Seki Kōwa (1642-1708) a Japanese, had a knowledge of determinants before 1683. They state that, while Leibnitz dealt with three equations only, Seki worked with n equations. Seki knew that a determinant of the nth order, when expanded, has n! terms; he knew also that rows and columns may be interchanged. Smith states that the Chinese had some idea of determinants even before Seki.

Gottfried Wilhelm Leibnitz (1646-1716) clearly had an idea of what is now known as the theory of determinants as shown by his letter to De L'Hospital which was dated April 28, 1693[3], but was not published until 1850. A manuscript bearing no date,

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1. Cajori, A history of mathematics, 80.

2. Smith, History of mathematics, I,440.

3. Smith, A source book in math., 267-269. Here is given a complete translation by Dr. Thomas F. Cope of the original letter.

but believed to date back to 1678, is apparently Leibnitz's first contribution. Cope's translated extract from this manuscript is:

> I have found a rule for eliminating the unknowns in any number of equations of the first degree, provided that the number of equations exceeds by one the number of unknowns. It is as follows:
>
> Make all possible combinations of the coefficients of the letters, in such a way that more than one coefficient of the same unknown and of the same equation never appear together. These combinations, which are to be given signs in accordance with the law which will soon be stated, are placed together, and the result set equal to zero will give an equation lacking all the unknowns.
>
> The law of signs is this: To one of the combinations a sign will be arbitrarily assigned, and the other combinations which differ from this one with respect to two, four, six, etc. factors will take the opposite sign; those which differ from it with respect to three, five, seven, etc. factors will of course take its own sign. For example, let
>
> $$10 + 11x + 12y = 0,$$
>
> $$20 + 21x + 22y = 0,$$
>
> $$30 + 31x + 32y = 0;$$
>
> there will result
>
> $$10\cdot21\cdot32 - 10\cdot22\cdot31 - 11\cdot20\cdot32$$
>
> $$+ 10\cdot22\cdot30 + 12\cdot20\cdot31 - 12\cdot21\cdot30 = 0.$$
>
> I consider also as coefficients those characters which do not belong to any of the unknowns, as 10, 20, 30.

Thus the contributions of Liebnitz are three in number: (1) A new notation, numerical in character and appearance, for individual members of an arranged group of magnitudes. (2) A rule for forming the terms of the expression which equated to zero is the result of eliminating the unknowns from a set of simple equations. (3) A rule for determining the signs of the terms in the said result; but this rule is expressed in an obscure way. Leibnitz's work is unimportant other than from a historical point of view in the development of the theory of determinants since his work was not published until the theory was advanced beyond what he knew of the subject.

Alexis Fontaine des Bertins[1] indicated a knowledge of the theory of determinants in his memoir of 1748. He apparently knew nothing of the work of Leibnitz. His work did not attract attention until after other mathematicians had made the theory of determinants popular.

The theory of determinants was rediscovered over half a century after Leibnitz's discovery, but this time the knowledge of the theory was kept alive in Europe. Thus in 1750 Cramer attracted the attention of other mathematicians in France to the theory of determinants which in time became the common property of all mathematicians. Cramer published a rule for finding the value of the unknowns in a system of n linear equations involving n unknowns. The rule may be divided into three parts:---(1) A rule for forming the terms of the common denominator of the fractions which express the values of the unknowns. (2) A rule for determining the sign of any individual term in the said common denominator. (3) A rule for obtaining the numerators from the expression for the common denominator.

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1. Muir, Theory of determinants, I,10.

Vandermonde (1771) was the first to recognize determinants as independent functions. He called them resultants. Because of this independent treatment and the many fundamental properties that he demonstrated, he is often called the real founder of the theory. He established eight new fundamental properties.

Laplace (1772) formulated the rule, which now bears his name, for the development of a determinant by its complementary minors; however, Vandermonde had already given a special case of it. This rule is very useful in the expansion of higher order determinants.

Lagrange (1773) added very little to the theory, but he was the first to apply determinants to problems other than elimination, principally in the domains of geometry and the theory of numbers.

November 30, 1812, was a very important day[1] in the development of the theory of determinants. Binet and Cauchy presented their memoirs this same day. The importance of their works probably exceed that of all that had been done before.

Binet's work centered on the development and uses of the multiplication theorem.

Cauchy defined many terms including determinants in the present sense; he made improvements on much of the earlier work.

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1. Muir, Theory of determinants, I,130.

Thomas Muir says of Cauchy's memoir:-

> On looking back, however, at Cauchy's memoir as a whole, one cannot but be struck with admiration both at the quality and the quantity of its contents. Supposing that none of its theorems had been new, and that it had not even presented a single old theorem in a fresh light, the memoir would have been most valuable, furnishing, as it did, to the mathematicians of the time an almost exhaustive treatise on the theory of general determinants. It is not too much to say, although it may come to many as a surprise, that the ordinary textbooks of determinants supplied to university students of the present day do not contain much more of the general theory than is to be found in Cauchy's memoir of about eighty years ago. One apparently trivial instrument, which Cauchy had not received from his predecessors and which he did not make for himself, viz., a notation of determinants whose elements had special values, is at the foundation of the whole difference between his treatise and those at present employed. When this want came to be supplied later on, the functions crept steadily into everyday use, and a fresh impetus was consequently given to the study of them. But if from the work of the said eighty years all researches regarding special forms of determinants be left out, and all investigations which ended in mere rediscoveries or in rehabilitations of old ideas, there is a surprisingly small proportion left. If one bears this in mind, and recalls the fact, temporarily set aside that Cauchy, instead of being a compiler, presented the entire subject from a perfectly new point of view, added many results previously unthought of, and opened up a whole avenue of fresh investigation, one cannot but assign to him the place of highest honour among all the workers from 1693 to 1812. It is, no doubt, impossible to call him, as some have done, the formal founder of the theory. This honour is certainly due to Vandermonde, who, however, erected on the foundation comparatively little of a superstructure. Those who followed Vandermonde contributed, knowingly or unknowingly, only a stone or two larger or smaller, to the building. Cauchy relaid the foundation, rebuilt the whole, and initiated new enlargements; the result being an edifice which the architects of to-day may still admire and find worthy of study.

The work from this time turned to the various special forms in determinants. The greatest contributor in this field was Jacobi. He used the functional determinant which Sylvester has called

_Jacobian_; he likewise treated that class of alternating functions which Sylvester has called _Alternates_. Jacobi was also the pioneer in _Skew determinants_ and _Orthogonants_. The next persons of importance in the field were Sylvester and Cayley, who added much to the theory of determinants by discovering and applying many new forms.

The use of determinants became so extensive that special textbooks were written on the subject. There[1] is not complete agreement as to which work shall be called the first special book on the subject. Some regard W. Spottiswoode's monograph of 63 pages, entitled _Elementary Theorems Relating to Determinants_, published in London in 1851, as the first. Others are inclined to regard the more extensive and more accurate Italian work by F. Brioschi, published in Pavia in 1854 under the title _Da Teorica dei Determinanti, e le sue Principali Applicazioni_, as the first published book on this subject. This book was translated into other languages. Other books were written in the next few years.

Thomas Muir found, while gathering material for his _Treatise on the Theory of Determinants_, published in 1880, that there were many inaccurate statements in regard to the authorship and history of many theorems. He therefore resolved to collect the titles and to determine the authors of all the writings which had appeared on the theory of determinants up to 1881. His find-

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1. Miller, _Historical introduction to math. lit._, 192.

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ings were published under the title A List of Writings on Determinants. He has continued this work and has brought it up to 1923.

Using these lists of writing as a basis Muir has written a set of four volumes entitled The Theory of Determinants in the Historical Order of Development up to 1900. In a later book he has brought the history up to 1920.

In the last sixty years a very large amount of work has been done on determinants; some of the most outstanding contributers of the period are Scott, Hanus, Bôcher, and Stouffer.

The foregoing sketch is merely a brief account of the more important points in the history of determinants. Since the advent of determinants, many mathematicians have done work in this field. There are two reasons for this great interest in determinants: (1) They are a very interesting subject as pure mathematics; (2) many persons have extended the theory of determinants to make it applicable to their special problems.

# CHAPTER III

## GENERAL PROPERTIES

The first question that arises is, what is the theory of determinants? J. J. Sylvester says[1],

> It is an algebra upon algebra; a calculus which enables us to combine and foretell the results of algebraic operations, in the same way as algebra itself enables us to dispense with the performance of the special operations of arithmetic. All analysis must ultimately clothe itself in this form.

Scott and Mathews[2] say,

> Determinants are algebraical expressions of a particular type calculated by a systematic rule and expressed by a special notation.

This important class of algebraic functions owes its origin to an attempt to formulate the solutions of general systems of simultaneous linear equations. Such a system of the second order is

$$a_1x + b_1y = k_1,$$

$$a_2x + b_2y = k_2.$$

Multiply the first equation by $b_2$, the second by $-b_1$, and add the resulting equations. One obtains

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1. Merrill and Smith, First course in higher algebra, 28.

2. Scott and Mathews, Theory of determinants, 1.

$$(a_1b_2 - a_2b_1)x = k_1b_2 - k_2b_1.$$

In a similar manner the equation in y is

$$(a_1b_2 - a_2b_1)y = a_1k_2 - a_2k_1.$$

It is seen that x and y have the common multiplier,

$$a_1b_2 - a_2b_1,$$

which may be written in the symbolic form

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},$$

This quantity is called a determinant of the second order. It is also called the determinant of the coefficients. The letters $a_1$, $b_1$, $a_2$, etc., are called the elements of the determinant. The products $a_1b_2$ and $a_2b_1$ are called the terms of the expansion. The values of x and y may now be expressed in the form

$$x = \frac{\begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

Consider now the solution of a system of the third order,

$$a_ix + b_iy + c_iz = k_i, \qquad (i = 1, 2, 3).$$

Multiply the first, second and third equations by

$$b_2c_3 - b_3c_2, \quad b_3c_1 - b_1c_3, \quad b_1c_2 - b_2c_1,$$

respectively, and add the resulting equations;

$$(a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3)x =$$

$$k_1b_2c_3 + k_2b_3c_1 + k_3b_1c_2 - k_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3.$$

The value of x becomes

$$x = \frac{\begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

These are determinants of the third order. The values of y and z, found in a similar manner, are

$$y = \frac{\begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

One will notice that all the denominators are the same and that the numerator of that unknown whose value is sought is obtained by replacing the coefficients of the unknown by the corresponding constant terms. A convenient rule, due to Sarrus[1], for the expansion of any determinant of the third order is: Consider the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Alongside of this repeat the first and second columns in order

$$\begin{array}{ccccc} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{array}$$

and form the product of each set of three elements lying in lines parallel to the diagonals of the original array. Those three which lie in lines descending from left to right have the positive sign, the other three products are negative. Accordingly the determinant is

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3.$$

A modification of the above process saves much time when one understands the procedure. Let us consider the same determinant,

$$\begin{array}{ccc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array}.$$

1. Scott and Mathews, Theory of determinants, 3.

The products are formed along the lines and the same rules of signs hold. It is seen that each product always contains three terms.

Before taking up the expansion of a determinant of order n it will be well to discuss some properties of permutations.

Any series of n elements $a_1$, $a_2$, $\ldots a_n$, arranged in order according to the magnitude of the numbers forming the suffixes is called the natural or original order of the letters. Any other order is called a permutation of the letters or elements. When any element precedes another element with a smaller suffix, it is known as an inversion[1].

Permutations are usually divided into two classes; the first class contains those permutations which have an even number of inversions, the second those which have an odd number. The first class is often called the positive and the second the negative permutation. Thus the interchanging of any two elements of a permutation changes its class. If in any arrangement, each suffix is subtracted from all that follow it and the differences are multiplied together, the sign of the product will depend on the number of inversions in this arrangement, the sign being positive if the number of inversions is even and negative if the number of inversions is odd. The number of permutations of n elements, taken n at a time, is equal to n!. The number of positive permutations of n elements is always equal to the

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1. Scott and Mathews, Theory of determinants, 8.

number of negative permutations.

The $n^2$ elements arranged is the square array

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots a_{2n} \\ a_{31} & a_{32} & a_{33} \cdots a_{3n} \\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ a_{n1} & a_{n2} & a_{n3} \cdots a_{nn} \end{vmatrix}$$

is called a determinant of order n. Here the double suffix is used; the first suffix gives the number of the row and the second the number of the column in which the element lies. Thus the element $a_{ij}$ is in the ith row and jth column. The diagonal line containing $a_{11}\ a_{22}\ a_{33} \cdots a_{nn}$ is called the leading or principal diagonal, the position occupied by it is the leading position.

The expansion of the above determinant is obtained by holding either the first, or second, suffix of each element of the principal diagonal fixed and permuting the other. To those of the n! resulting terms that involve the positive permutations give the plus sign; to those involving the negative permutations, the minus sign. The algebraic sum of these terms is the value of the determinant.

The expansion may be stated in a different manner. Form all possible products of n elements each taking one and only one element from each row and each column. If the elements of each term are arranged in order of columns (rows) find the number of exchanges of elements needed to bring them into the order of the rows (columns) and give to each term the positive or negative sign according as this number of exchanges is even or odd. The algebraic sum of these terms is the value of the determinant.

It is not important whether the first or second suffix is held constant and the other permuted. In either case there will be n! terms and the sign of each term will be the same since the number of inversions will be even or odd as before.

On interchanging rows and columns in a determinant of nth order, one gets

$$\begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \cdot & \cdot & \cdot & \cdot \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix}.$$

This is different from the original only in that the suffixes of each element are interchanged. In the expansion of the new determinant the same result is obtained since the determinant may be expanded by permuting either suffix. Thus the value of a determinant is not altered by interchanging the rows and columns. Therefore any property true for the rows of a determinant is also true for the columns and visa verca. The term <u>line</u> is used to mean either row or column or both.

If any two rows or columns are interchanged the value of the determinant is altered only in sign. For, interchanging two lines is the same as interchanging in each term of the expansion, the suffixes corresponding to these lines. This changes the sign of each term; therefore, the sign of the whole determinant is reversed.

If each element of a line is multiplied or divided by a common factor, it is equivalent to performing the same operation on the determinant as a whole; for each term of the expansion contains a single element from the given line. The common operation thus is performed once and only once on each term of the expansion, and the determinant is, therefore, multiplied or divided by that factor.

If two rows or columns of a determinant are identical, it is equal to zero. For interchanging these two lines would reverse the sign of the determinant, but its value is not altered since these lines are identical; and the only number that is equal to its negative value is zero.

If the corresponding elements of two rows or columns have a common ratio, the determinant vanishes. This is evident since one of these lines could be multiplied by the common ratio thus making two identical lines.

A determinant having a line whose elements are each the sum of two quantities can be expressed as a sum of two determinants.

Let

$$D = \begin{vmatrix} a_1 + b_1 & c_1 & d_1 \cdots \\ a_2 + b_2 & c_2 & d_2 \cdots \\ a_3 + b_3 & c_3 & d_3 \cdots \\ \cdots & \cdots & \cdots \end{vmatrix}$$

be such a determinant. Then the terms of the expansion will contain the element $a_i + b_i$ which may be broken up into $a_i$ times the other elements of the term and $b_i$ times the same. Thus these separate terms will make up the two determinants

$$\begin{vmatrix} a_1 & c_1 & d_1 \cdots \\ a_2 & c_2 & d_2 \cdots \\ a_3 & c_3 & d_3 \cdots \\ \cdots & \cdots & \cdots \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} b_1 & c_1 & d_1 \cdots \\ b_2 & c_2 & d_2 \cdots \\ b_3 & c_3 & d_3 \cdots \\ \cdots & \cdots & \cdots \end{vmatrix}$$

whose sum is equal to D.

The value of a determinant is not changed if to the elements of any row the products of the corresponding elements of another row by the same arbitrary constant is added. Consider the determinant D:

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Let the first column be increased by x times the second. Then one has

$$\begin{vmatrix} a_1 + xb_1 & b_1 & c_1 \\ a_2 + xb_2 & b_2 & c_2 \\ a_3 + xb_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + x \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix}$$

but the last determinant is zero. This theorem is often used to combine the rows and columns in such a manner to make as many of the elements of a determinant zero as possible, before the determinant is expanded.

The expansion of determinants of higher order is quite lengthy; these higher order determinants are often expanded by minors or cofactors.

The determinant[1] of order n-1 obtained by removing the ith row and the jth column of a determinant of order n is called the minor of the element $a_{ij}$. A determinant D of order n may be expanded according to the elements of any row or column.

In

$$D = \begin{vmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{n1} & a_{n2} \cdots a_{nn} \end{vmatrix}$$

denote the minor of any element by the corresponding capital letter so that $a_{ij}$ has the minor $A_{ij}$. Then if the determinant is expanded

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1. Dickson, Theory of education, 109.

by the first row it is necessary to prove that,

$$D = a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13} \cdots \pm a_{1n}A_{1n}.$$

The sign of $a_{ij}$ is plus or minus according to whether the sum of the suffixes is even or odd.

Now $A_{11}$ is a determinant of order n-1 and when it is expanded and multiplied by $a_{11}$, one obtains all the terms of D which contain $a_{11}$ with the proper sign. The same will be true for $A_{12}$, $A_{13}$, ... $A_{1n}$. Upon taking the algebraic sum of these n expansions, $n[(n-1)!]$ or n! terms are obtained containing each n elements. Thus our theorem is proved.

In a determinant of order n, those minors of order n-1 are called <u>first minors</u>; those of order n-2 are called <u>second minors</u>; and those of order n-r are called <u>rth minors</u>.

Minors formed by the deletion of the same row and column are called <u>principal minors</u>. Principal minors are sometimes called <u>co-axial minors</u>.

The cofactor, $E_{ij}$, of the element $a_{ij}$ is the signed minor of $a_{ij}$:

$$E_{ij} = (-1)^{i+j}A_{ij}.$$

Thus the sum of the elements of a row or column multiplied by their cofactors is equal to the determinant. Expansion by cofactors instead of minors has the advantage that the sign is always positive, in other words the sign is intrinsic.

If each element[1] of a row, or column, of a determinant is multiplied by the cofactor of the corresponding element of a different row, or column, the sum of the resulting products is zero. Consider the three row determinant,

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

The multiplication of the elements of column one by the cofactors of the corresponding elements of column two gives,

$$a_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + a_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}.$$

Now write this as a three row determinant and it is evident that the determinant vanishes,

$$\begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix}.$$

Any rth minor of a given determinant and the determinant of the $r^2$ elements at the intersection of the rows and columns deleted in forming it are called, with respect to each other, complementary minors. The determinant[2]

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1. Grausten, Itro. to higher geometry, 14.

2. Dickson, Theory of equations, 122-125.

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

has as two-rowed complementary minors

$$M = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad M' = \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix}, \quad \text{etc.;}$$

since either is obtained by removing from D all the rows and columns having an element which occurs in the other. Any element may be regarded as a one-rowed minor and is complementary to its minor.

Laplace established the theorem for developing a determinant by its complementary minors. Any determinant D is equal to the sum of all the signed products MM', where M is an r-rowed minor having its elements in the first r columns of D, and M' is the minor complementary to M, while the sign is plus or minus according as an even or odd number of interchanges of rows of D will bring M into the position occupied by the minor $M_1$ whose elements lie in the first r rows and first r columns of D.

The case for $r = 1$ consists of development by first row for which proof has been given.

When D is of rank n, then

$$M_1 = \begin{vmatrix} a_{11} & a_{12} \cdots a_{1r} \\ a_{21} & a_{22} \cdots a_{2r} \\ \cdots & \cdots \\ a_{r1} & a_{r2} \cdots a_{rr} \end{vmatrix} \qquad M_1' = \begin{vmatrix} a_{r+1r+1} \cdots a_{r+1n} \\ a_{r+2r+1} \cdots a_{r+2n} \\ \cdots \\ a_{nr+1} \cdots a_{nn} \end{vmatrix}.$$

Any term of product $M_1 M_1'$ is of the type

$$(-1)^i a_{i_1 1} a_{i_2 2} \cdots a_{i_r r} (-1)^j a_{i_{r+1} r+1} \cdots a_{i_n n},$$

where $i_1, \ldots, i_r$ is an arrangement of first r suffixes derived from original order 1, 2, 3, ..., r by i interchanges. Hence $i_1, \ldots, i_n$ is an arrangement of 1, ..., n derived by $i+j$ interchanges, so that the terms of the product $M_1 M_1'$ are terms of D with the proper sign.

It can now be shown by the theorem on interchanging of two rows or columns that any term of any of the products $\pm MM'$ is a term of D. No term will appear twice, and when all the products of MM' are taken all terms of D will be included.

Consider another method of expanding determinants[1], which in the simplified and rather mechanical form each step is not equal to the others, since certain terms are left out. The simplified process is: A determinant D of order n may be converted into a determinant D' of order n-1 whose elements are two row

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1. The proof of this method has been developed by the author of this thesis; however, it has been learned recently that Dr. A. C. Aitken, F. R. S. published a proof in Transactions of the Faculty of Actuaries, vol. XIII, 1931.

minors of D; likewise D' may be converted into a determinant D" of order n-2 whose elements are two row minors of D' divided by the common elements of D that have now been included too many times. This reduction is continued until a determinant of order two is obtained which may be expanded by the ordinary method.

Let n equal 3, then

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \frac{\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix}}{a_{22}} = \frac{\begin{vmatrix} a_{11} a_{12} \\ a_{21} a_{22} \end{vmatrix}\begin{vmatrix} a_{22} a_{23} \\ a_{32} a_{33} \end{vmatrix} - \begin{vmatrix} a_{12} a_{13} \\ a_{22} a_{23} \end{vmatrix}\begin{vmatrix} a_{21} a_{22} \\ a_{31} a_{32} \end{vmatrix}}{a_{22}}$$

$$D = a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - \frac{a_{12}a_{21}a_{23}a_{32}}{a_{22}} + \frac{a_{12}a_{21}a_{23}a_{32}}{a_{22}} + a_{21}a_{32}a_{13}$$

$$-a_{13}a_{22}a_{31} + a_{31}a_{12}a_{23} - a_{12}a_{21}a_{33},$$

which is the correct value of D.

The method can be illustrated better for a higher ordered determinant. Consider an example when n is equal to five;

$$D = \begin{vmatrix} a_{11} a_{12} a_{13} a_{14} a_{15} \\ a_{21} a_{22} a_{23} a_{24} a_{25} \\ a_{31} a_{32} a_{33} a_{34} a_{35} \\ a_{41} a_{42} a_{43} a_{44} a_{45} \\ a_{51} a_{52} a_{53} a_{54} a_{55} \end{vmatrix} \rightarrow \begin{vmatrix} B_{11} B_{12} B_{13} B_{14} \\ B_{21} B_{22} B_{23} B_{24} \\ B_{31} B_{32} B_{33} B_{34} \\ B_{41} B_{42} B_{43} B_{44} \end{vmatrix} \rightarrow \begin{vmatrix} C_{11} C_{12} C_{13} \\ C_{21} C_{22} C_{23} \\ C_{31} C_{32} C_{33} \end{vmatrix} \rightarrow \begin{vmatrix} E_{11} E_{12} \\ E_{21} E_{22} \end{vmatrix}$$

$$D = \frac{E_{11}E_{22} - E_{12}E_{21}}{C_{22}},$$

where

$$B_{11} = \begin{vmatrix} a_{11} a_{12} \\ a_{21} a_{22} \end{vmatrix}, \quad B_{12} = \begin{vmatrix} a_{12} a_{13} \\ a_{22} a_{23} \end{vmatrix}, \quad B_{ij} = \begin{vmatrix} a_{ij} & a_{i\ j+1} \\ a_{i+1j} & a_{i+1j+1} \end{vmatrix},$$

$$C_{11} = \begin{vmatrix} B_{11} B_{12} \\ B_{21} B_{22} \end{vmatrix} \frac{1}{a_{22}} \quad C_{ij} = \begin{vmatrix} B_{ij} & B_{ij+1} \\ B_{i+1j} & B_{i+1j+1} \end{vmatrix} \frac{1}{a_{i+1j+1}},$$

$$E_{ij} = \begin{vmatrix} C_{ij} & C_{ij+1} \\ C_{i+1j} & C_{i+1j+1} \end{vmatrix} \frac{1}{B_{i+1j+1}}.$$

The justification of this method is based entirely upon a corollary of Bocher's[1], namely:

> If D is any determinant, and S is the second minor obtained from it by striking out its ith and kth rows and its jth and lth columns, and if we denote by $A_{ij}$ the cofactor of the element which stands in the jth column and ith row of D, then
>
> $$\begin{vmatrix} A_{ij} & A_{il} \\ A_{kj} & A_{kl} \end{vmatrix} = (-1)^{i+l+j+k} DS.$$

Now when i and j are equal to one and k and l are equal to n there sum will be even. Then

$$D = \begin{vmatrix} A_{ij} & A_{il} \\ A_{kj} & A_{kl} \end{vmatrix} \frac{1}{S}.$$

Thus in the case of n equals 5:

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1. Bôcher, Higher algebra, 33.

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} = \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} & a_{14} & a_{15} \\ a_{22} & a_{23} & a_{24} & a_{25} \\ a_{32} & a_{33} & a_{34} & a_{35} \\ a_{42} & a_{43} & a_{44} & a_{45} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{51} & a_{52} & a_{53} & a_{54} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} & a_{24} & a_{25} \\ a_{32} & a_{33} & a_{34} & a_{35} \\ a_{42} & a_{43} & a_{44} & a_{45} \\ a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} \end{vmatrix} \frac{1}{\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}};$$

now break up the fourth order determinants into third order, and these into second order determinants, using the same notation for B's as above, D now equals:

$$D = \begin{vmatrix} \begin{vmatrix} \frac{\begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{vmatrix}}{a_{22}} & \frac{\begin{vmatrix} B_{12} & B_{13} \\ B_{22} & B_{23} \end{vmatrix}}{a_{23}} \\ \frac{\begin{vmatrix} B_{21} & B_{22} \\ B_{31} & B_{32} \end{vmatrix}}{a_{32}} & \frac{\begin{vmatrix} B_{22} & B_{23} \\ B_{32} & B_{33} \end{vmatrix}}{a_{33}} \end{vmatrix} \frac{1}{B_{22}} & \frac{1}{B_{23}} \begin{vmatrix} \frac{\begin{vmatrix} B_{12} & B_{13} \\ B_{22} & B_{23} \end{vmatrix}}{a_{23}} & \frac{\begin{vmatrix} B_{13} & B_{14} \\ B_{23} & B_{24} \end{vmatrix}}{a_{24}} \\ \frac{\begin{vmatrix} B_{22} & B_{23} \\ B_{32} & B_{33} \end{vmatrix}}{a_{33}} & \frac{\begin{vmatrix} B_{23} & B_{24} \\ B_{33} & B_{34} \end{vmatrix}}{a_{34}} \end{vmatrix} \\ \begin{vmatrix} \frac{\begin{vmatrix} B_{21} & B_{22} \\ B_{31} & B_{32} \end{vmatrix}}{a_{32}} & \frac{\begin{vmatrix} B_{22} & B_{23} \\ B_{32} & B_{33} \end{vmatrix}}{a_{33}} \\ \frac{\begin{vmatrix} B_{31} & B_{32} \\ B_{41} & B_{42} \end{vmatrix}}{a_{42}} & \frac{\begin{vmatrix} B_{32} & B_{33} \\ B_{42} & B_{43} \end{vmatrix}}{a_{43}} \end{vmatrix} \frac{1}{B_{32}} & \frac{1}{B_{33}} \begin{vmatrix} \frac{\begin{vmatrix} B_{22} & B_{23} \\ B_{32} & B_{33} \end{vmatrix}}{a_{33}} & \frac{\begin{vmatrix} B_{23} & B_{24} \\ B_{33} & B_{34} \end{vmatrix}}{a_{34}} \\ \frac{\begin{vmatrix} B_{32} & B_{33} \\ B_{42} & B_{43} \end{vmatrix}}{a_{43}} & \frac{\begin{vmatrix} B_{33} & B_{34} \\ B_{43} & B_{44} \end{vmatrix}}{a_{44}} \end{vmatrix} \end{vmatrix} \frac{a_{33}}{\begin{vmatrix} B_{22} & B_{23} \\ B_{32} & B_{33} \end{vmatrix}}.$$

It will be seen now that this process and the one above are closely related. Note

$$E_{11} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \begin{vmatrix} \frac{\begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{vmatrix}}{a_{22}} & \frac{\begin{vmatrix} B_{12} & B_{13} \\ B_{22} & B_{23} \end{vmatrix}}{a_{23}} \\ \frac{\begin{vmatrix} B_{21} & B_{22} \\ B_{31} & B_{32} \end{vmatrix}}{a_{32}} & \frac{\begin{vmatrix} B_{22} & B_{23} \\ B_{32} & B_{33} \end{vmatrix}}{a_{33}} \end{vmatrix} \frac{1}{B_{22}}$$

$E_{12}, E_{21}, E_{22}$ are equal respectively to each of the other four row determinants. The C's are equal to the elements of the determinants E. Thus it is evident that this method of expanding determinants can be extended to any order since it is based on Bôcher's corollary, which is quite general.

This method of expanding determinants breaks down if any of the divisors are zero but this often may be overcome by interchanging rows or columns.

A similar method of expanding determinants is[1]:

> If the first pair of elements in the first row of a determinant be taken in succession with every pair below it, and the determinants of the second order which have these pairs for rows be placed in order as the elements of the first column of a new determinant, and if the like be done in the case of the second and following pairs of consecutive elements in the row, then the new determinant thus obtained divided by the product of all the elements of the first row of the original determinant except the first and last is equal to the original determinant.

$$D = \begin{Vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \cdots & \begin{vmatrix} a_{1n-1} & a_{1n} \\ a_{2n-1} & a_{2n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & & \begin{vmatrix} a_{1n-1} & a_{1n} \\ a_{3n-1} & a_{1n} \end{vmatrix} \\ \cdots & \cdots & \cdots & \cdots \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{n2} & a_{n3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{1n-1} & a_{1n} \\ a_{nn-1} & a_{nn} \end{vmatrix} \end{Vmatrix} \frac{1}{a_{12}a_{13}a_{14}\cdots a_{1n-1}}$$

This method also breaks down if any of the divisors are

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1. A proof of this method will be found in Muir and Metzler's <u>Theory of determinants</u>, 63-66

zero, but this may be avoided by certain transformations.

The product of two determinants of the same order is equal to a determinant of like order in which the element of the ith row and jth column is the sum of the products of the elements of the ith row multiplied by the corresponding elements of the jth column.

The multiplication theorem may be stated also in this different form but it involves the same process[1]:

> Connect by plus signs the elements of each row of the first determinant D', likewise the elements of each column of the second determinant D". Then place the first row of D' upon each column of D" in turn and let each two elements as they touch become products. This will be the first row of the product determinant D. Perform the same operation with the second row of D' which gives the second row of D. This process is continued for the n rows of D' to form D.

One can justify these two methods of multiplication by Laplace's development, as is illustrated below, when n = 3:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ -1 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & -1 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & -1 & b_{31} & b_{32} & b_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} .$$

In the determinant of order 6, add to the elements of the fourth, fifth, and sixth columns the products of the elements of

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1. Weld, "Determinants" in Encyclopedia Americana, IX,21.

the first column by $b_{11}, b_{12}, b_{13}$, respectively. Next, add to the elements of the last three columns the products of the elements of the second column by $b_{21}, b_{22}, b_{23}$, respectively. Finally, add to the elements of the last three columns the products of the elements of the third column by $b_{31}, b_{32}, b_{33}$. The new determinant is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} \\ a_{21} & a_{22} & a_{23} & a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} \\ a_{31} & a_{32} & a_{33} & a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} & a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33} \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{vmatrix}.$$

By Laplace's development, this is equal to the 3-rowed minor of the upper-right corner. Therefore, this minor is equal to the product of $|a_{1n}|$ and $|b_{1n}|$.

# CHAPTER IV

## SPECIAL FORMS OF DETERMINANTS

### COMPOUND DETERMINANTS

A determinant with elements which are themselves determinants is called a compound determinant.

Sylvester[1] was very prominent in the early developments of compound determinants. Compound determinants have many applications. It will be noted that they have been used in the last two methods of expanding general determinants. A very important class of compound determinants is that in which each element is the cofactor of the corresponding element in another determinant, such a determinant is called the determinant adjugate to that other, or more simply just the adjoint.

Some important and interesting theorems on adjoint determinants are:

> If D' is the adjoint[2] of any determinant D, and M and M' are the corresponding m-rowed minors of D and D' respectively, then M' is equal to the product of $D^{m-1}$ by the algebraic complement of M.

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1. Muir, Theory of determinants, II, 192.

2. Bôcher, Algebra, 31.

If D is any determinant of the nth order and D' its adjoint, then

$$D' = D^{n-1}$$

From this property, the truth of the following definition is apparent. The determinant whose elements are those of the adjoint each divided by D is the <u>reciprocal</u> of D.

ALTERNANTS

Any determinant which is an alternating function is called an <u>alternant</u>: for example

$$\begin{vmatrix} 1 & a & a^2 & a(b^4c^4+b^4d^4+c^4d^4) \\ 1 & b & b^2 & b(c^4d^4+c^4a^4+d^4a^4) \\ 1 & c & c^2 & c(d^4a^4+d^4b^4+a^4b^4) \\ 1 & d & d^2 & d(a^4b^4+a^4c^4+b^4c^4) \end{vmatrix}$$

Cauchy[1] (1812) laid a good foundation for the study and development of the theory of this special type of determinant.

The product or quotient of two alternating functions of order n is a symmetric function of the same order.

The conditions for the identity of two alternating functions are: (1) that all the terms of the first function be contained in the second; (2) that the terms have the same numerical coefficients in both; (3) that one of the terms of the first has the same sign as the corresponding term of the second.

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1. Muir, <u>Theory of determinants</u>, I, 306.

Every alternant[1] of the nth order is evidently a function of n variables. To interchange two of these would be the same as to interchange two of the rows or columns of the determinant, and therefore would have the effect of merely changing the sign of the function; and as a function having this property is known as an alternating function, the origin of name alternant is apparent.

Every alternant with rational integral elements contain as a factor the difference-product of its variables.

SYMMETRIC DETERMINANTS

In a determinant there may be three kinds of symmetry[2]: (1) symmetry with respect to the principal diagonal, (2) symmetry with respect to the secondary diagonal, and (3) symmetry with respect to the center. (2) is closely related to (1) since by reversing the order of the rows and columns symmetry with respect to the secondary diagonal may be changed to symmetry with respect to the principal diagonal.

These three imply that: (1) $a_{ij} = a_{ji}$, that is, conjugate elements are equal, (2) $a_{ij} = a_{n+1-j,n+1-i}$, (3) $a_{ij} = a_{n+1-i,n+1-j}$, respectively.

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1. Muir and Metzler, Theory of determinants, 321.

2. Ibid., 18-19.

When (1) or (2) is true the determinant is said to be <u>axi-symmetric</u>, when (3) is true it is said to be <u>centrosymmetric</u>. When two of the three types exist at once the determinant is said to be <u>bisymmetric</u>.

Axisymmetric determinants[1] are first found in Lagrange's memior, 1773; however, he apparently did not recognize them as a special type. Their properties and theorems were much studied by Binet, Jacobi and Cauchy.

Some special properties[2] of axisymmetric determinants are: (1) conjugate lines are alike, (2) coaxial minors are axisymmetric, (3) conjugate minors are equal, (4) all compounds of the original are axisymmetric.

From the law of multiplication it follows that any even power of any determinant is expressible as an axisymmetric determinant.

Any power of an axisymmetric determinant is expressible as an axisymmetric determinant.

Any power of any determinant of the second order is expressible as an axisymmetric determinant.

If in an axisymmetric determinant the sum of the elements in every row is zero, then all the primary minors are numerically equal.

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1. Muir, <u>Theory of determinants</u>, I, 289.

2. Muir and Metzler, <u>Theory of Determinants</u>, 364-372.

Centrosymmetric Determinants

The determinant is the same, term by term, when read backwards as when read forwards.

Every centrosymmetric determinant D of even order 2m is expressible as the product of two determinants each of order m.

Any determinant of order n having the array of its last (n-1) rows centrosymmetric is expressible as the product of two determinants.

A determinant is said to be skew-centrosymmetric when every constituent is the negative of its conjugate with respect to the center. One of odd order would therefore have its center element zero.

Every skew-centrosymmetric determinant of even order is expressible as the difference of two squares.

Every skew-centrosymmetric determinant of odd order is equal to zero.

Skew Determinants

A determinant having conjugate elements equal but opposite in sign (i.e. $a_{ij} = -a_{ji}$) is called a skew determinant; and if in addition $a_{ii} = 0$ it is called a zero-axial skew determinant. Zero-axial skew determinants are sometimes called skew-symmetric.

In the case of zero-axial skew determinants it appears that

(1) coaxial minors are zero-axial skew; (2) conjugate minors are equal or differ only in sign, according as they are of even or of odd order; (3) the adjoint determinant is skew if of even order, and axisymmetric if of odd order; (4) the determinant of odd order vanishes; (5) the adjoint of an even order determinant is a zero-axial.

The first mathematician to make a definite reference to this type of a determinant seems to have been Jacobi[1]. He considered these new functions as separate from and independent of determinants[1].

## Persymmetric Determinants[2]

A determinant such that each line perpendicular to the principal diagonal has all its elements alike is called a persymmetric determinant. In the persymmetric determinant of the nth order there are at most 2n-1 distinct elements, viz., those of the principal diagonal and one adjacent minor diagonal. This type of a determinant is also called orthosymmetrical[3] by some authors.

## Circulants

E. Catalan's[4] paper of the year 1846 contained several ex-

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1. Muir, Theory of determinants, I, 395.
2. Muir and Metzler, Theory of determinants, 419.
3. Scott and Mathews, Theory of determinants, 99.
4. Muir, Theory of determinants, II, 401.

amples of what is now known as circulant determinants.

A determinant[1] such that any row is got from the preceding row by passing the last element over the others to the first place is called a circulant. The circulant whose first row is $a_1, a_2 \ldots a_n$, will have the second row $a_n, a_1 \ldots a_{n-1}$, etc. It is usually denoted by C.

It is frequently more convenient to define circulant as a determinant such that any row is got from the preceding row by passing the first element over the others to the last place. This circulant is denoted by C'.

The determinant formed by changing the signs of all the elements on one side of the principal diagonal of a circulant C is called a skew circulant. For it the functional symbol SC is used.

A circulant C' is evidently a persymmetric determinant with but n distinct elements.

A peculiar property[2] of the circulant C is that it is divisible by

$$a_1 + a_2 w + a_3 w^2 + \ldots + a_n w^{n-1},$$

where w is a root of the equation $x^n = 1$.

Compound circulant determinants are called block circulants.

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1. Muir and Metzler, Theory of determinants, 442.
2. Scott and Mathews, Theory of determinants, 102.

CONTINUANTS

It is doubtful if the connection between continued fractions and determinants was considered before 1853. In that year J. J. Sylvester[1] published a paper in which he showed how a continued fraction could be expressed in terms of a determinant. A <u>continuant</u>[2] is a determinant all of whose elements are zero except those in the main diagonal and in the two adjacent diagonal lines parallel to and on either side of the main diagonal.

A continuant of order n is:

$$\begin{vmatrix} a_1 & b_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_1 & a_2 & b_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & c_2 & a_3 & b_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & c_{n-1} & a_n & \cdot \end{vmatrix}$$

Consider the system of equations[3]

$$x = a_1x_1 + b_1x_2$$
$$0 = c_1x_1 + a_2x_2 + b_2x_3$$
$$0 = c_2x_2 + a_3x_3 + b_3x_4$$
$$\cdots\cdots\cdots\cdots$$

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1. Muir, <u>Theory of determinants</u>, II, 413.
2. Muir and Metzler, <u>Theory of determinants</u>, 516.
3. Scott and Mathews, <u>Theory of determinants</u>, 213.

then

$$\frac{x_1}{x} = \frac{1}{a_1 + \frac{b_1 x_2}{x_1}}, \qquad \frac{x_2}{x_1} = \frac{-c_1}{a_2 + \frac{b_2 x_3}{x_2}}, \qquad \ldots$$

Hence $\frac{x_1}{x}$ is a continued fraction.

To determine the nth convergent, i.e., the value of the fraction when one stops at $\frac{b_n}{a_n}$, one must suppose that $x_{n+1}$ and all succeeding x's vanish, whence one has the system of equations

$$\begin{aligned} x &= a_1 x_1 + b_1 x_2 \\ 0 &= c_1 x_1 + a_2 x_2 + b_2 x_3 \\ 0 &= c_2 x_2 + a_3 x_3 + b_3 x_4 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ 0 &= c_{n-1} x_{n-1} + a_n x_n . \end{aligned}$$

Solving this set of equations, $\frac{x_1}{x}$ becomes a continued fraction which is expressible as a continuant.

Some special properties of continuants are (1) no term of the continuant can contain two consecutive b's or c's; (2) if a term contains $b_i$ it also contains $c_i$; (3) no term of the continuant can be formed in which an odd number of consecutive a's are omitted; (4) one term of the continuant is obviously $a_1 a_2 \ldots a_n$, other terms can be formed from this term by replacing any pair of consecutive a's by the product of the b and c having the same suffix as the first a of the pair with a negative sign (for example, $a_i a_{i+1}$ may be replaced by $-b_i c_i$); (5) all terms of the continuant are obtained by using the process (4) to the full extent.

## ORTHOGONANTS

This special form of determinant is connected with a problem in coordinate geometry--the problem of transformation from one set of coordinate axes to another set having the same origin. A determinant is called an orthogonant when it is the determinant of an orthogonal substitution.

Here again Jacobi (1827)[1] was a pioneer. The problem which he wished to solve was to transform an expression of the form

$$Ax^2 + By^2 + Cz^2 + 2ayz + 2bzx + 2cxy,$$

where x, y, z, are the coordinates of a point referred to an oblique coordinate-system, into an expression of the form

$$Lu^2 + Mv^2 + Nw^2,$$

where u, v, w are the coordinates of the same point referred to a rectangular system having the same origin. Thus the things directly sought were the nine coefficients which give each of the original coordinates in terms of the new.

Jacobi at this time apparently did not realize the importance of this new form of determinants; since in the same year he published another paper on the transformation of a double integral in which the use of orthogonants should have shown this problem to be closely related to the former problem.

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1. Muir, Theory of determinants, I, 410.

The square of an orthogonant[1] is equal to unity. It follows that the orthogonant is equal to $\pm 1$. An orthogonant having the value $+1$ is said to be proper and one which has the value $-1$ is said to be improper.

## JACOBIANS

It is likely that determinants in which the number of a row is distinguished by differentiation with respect to a definite variable, and in which the number of a column is distinguished by a particular function set for differentiation, may have appeared before the time of Jacobi. There can be little doubt that expressions like

$$\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{y}\cdot\frac{\partial v}{\partial x}$$

had appeared many times. It remained for Cauchy[2], 1815, to extend the process to a three row determinant. Jacobi made a very thorough study of this new use of determinants which was given his name by Sylvester.

If there be n functions[3] all of the same n variables, the determinant which in every case has the element in its ith row and

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1. Muir, and Metzler, Theory of determinants, 566.

2. Muir, Theory of determinants, I, 346-350.

3. Muir and Metzler, Theory of determinants, 635.

jth column equal to the differential coefficient of the ith function with respect to the jth variable is called the Jacobian of the set of functions with respect to the said variables.

The notations[1]

$$\frac{d(y_1,y_2,\ldots y_n)}{d(x_1,x_2,\ldots x_n)}; \qquad J(y_1,y_2,\ldots y_n)$$

are two of the common ones employed for Jacobians. The first renders evident the remarkable analogy between Jacobians and ordinary differential coefficients. The second is useful when there is no doubt as to the independent variables.

If the y's are explicit functions, the Jacobian is formed by direct differentiation.

If the functions $y_1,y_2,\ldots y_n$ are not independent, but are connected by an equation

$$f(y_1,y_2,\ldots y_n) = 0$$

the Jacobian vanishes. Thus if the Jacobian vanishes the functions are not independent, and conversely.

Jacobians are very useful in the study of advanced calculus, differential equations, and differential geometry.

HESSIANS

The Jacobian[2] of the partial differential coefficients of

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1. Scott and Mathews, Theory of determinants, 163.

2. Weld, "Determinants" in The Encyclopedia Americana, IX, 22.

a function, taken with respect to its several variables, is called the Hessian of the function. The Hessian is a symmetrical determinant. In symbols

$$H(u) = J\left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}\right) = \left|\frac{\partial^2 u}{\partial x_1^2}\ \frac{\partial^2 u}{\partial x_1^2} \cdots \frac{\partial^2 u}{\partial x_n^2}\right| .$$

If the given function is the ternary quadric

$$w \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy,$$

$$H(w) \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix};$$

a determinant, which in this case is also called the discriminant, for the reason that its vanishing is the condition that (w) be resolvable into linear factors.

The Jacobian and the Hessian are covariants. Both were originally called functional determinants. The general idea of the Hessian first occurred to Hesse[1] in 1843; however determinants of this form appeared earlier. A. Cayley and Jacobi soon took up this form after Hesse called attention to it.

If u is a function[2] of $x_1\ x_2, \ldots x_n$ and $u_i$ denotes $\partial u/\partial x_i$, and $u_{ij}$ denotes $\partial^2 y/\partial x_j \partial x_i$ and if

$$c_i u_i + c_2 u_2 + \ldots + c_n u_n = 0$$

where the c's are constant, then the Hessian of u vanishes and u may be transformed into a function with one less variable.

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1. Muir, Theory of determinants, II, 376.

2. Muir and Metzler, Theory of determinants, 376.

WRONSKIANS

This special form appeared for the first time in Wronski's[1] paper published in 1812. The name was suggested by Thomas Muir (1882) in his Textbook on Determinants.

If there be n functions[2] of one and the same variable x, the determinant which has in every case the elements in its ith row and jth column, the (i-1) differential coefficient of the jth function is called the <u>Wronskian</u> of the function with respect to x. The Wronskian of $y_1, y_2, y_3$ is denoted by $W_x(y_1 y_2 y_3)$.

The only non-vanishing term in the differential coefficient of a Wronskian is the one obtained by differentiating each element of the last row. Thus if $y_i(j)$ denotes the jth differential coefficient of $y_1$, then

$$W_x(y_1 y_2 \cdots y_n) \equiv |y_1 y_{2(1)} y_{3(2)} \cdots y_{n(n-1)}|$$

and

$$\frac{d}{dx} W_x(y_1 y_2 \cdots y_n) = |y_1 y_{2(1)} \cdots y_{n-1(n-2)} y_{n(n)}|$$

If a set of n functions of the same variable be connected by a linear relation with coefficients which are constant with respect to the variables the Wronskian of the functions vanishes.

Let the relation be

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0.$$

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1. Muir, <u>Theory of determinants</u>, II, 219.

2. Muir and Metzler, <u>Theory of determinants</u>, 662-671.

then

$$W(y_1, y_2, \ldots, y_n) = 0.$$

This is very important in the study and solution of systems of linear differential equations.

## BORDERED DETERMINANTS

If to a determinant[1] of the nth order one or more rows and the same number of columns of n quantities each are added and the vacant corner filled in with zeros, the resulting determinant is called a <u>bordered determinant</u>.

Sylvester[2] apparently was the first to use the expression bordered for this type of determinant; he did so in 1852. Others had recognized them as a special form but no one had suggested a name for them.

If a determinant of the nth order is bordered with n rows and n columns, the resulting determinant has a value which depends on the bordering quantities only. If a determinant of the nth order is bordered with more than n rows and columns, the resulting determinant always has the value zero. If a determinant of the nth order be bordered by p rows and p columns ($p<n$) of independent variables, the resulting determinant is a polynomial of degree 2p in the bordering quantities, whose coefficients are the pth minors of the original

1. Bôcher, <u>Algebra</u>, 28.
2. Muir, <u>Theory of determinants</u>, III, 432.

determinant; and conversely, every pth minor of the original determinant is the coefficient of at least one term of this polynomial.

# CHAPTER V

## APPLICATIONS

The first and most important use of determinants is in the solution of simultaneous linear equations. The method used in solving three equations has already been explained.

In the solution of a system of n linear equations in n unknowns, the same procedure is followed as in solving three equations in three unknowns. In the n equations:

$$a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \dots + a_{in}x_n = k_i, \qquad (i = 1,2, \dots n),$$

if the determinant D of the coefficients is zero and if at least one of the determinants $K_i$ of the numerators is not zero, then the equations are evidently inconsistent. If D and all the K's are zero, the former results give no information concerning the unknowns $x_i$, then one may resort to the following theorem[1]:

> Let the determinant D of the coefficients of the unknowns in the given n equations be of rank r, r<n. If the determinants K obtained from the (r+1)-rowed minors of D by replacing the elements of any column by the corresponding known terms $k_i$ are not all zero, the equations are inconsistent. But if these determinants K are all zero, the r equations involving the elements of a non-vanishing r-rowed minor of D determine uniquely r of the unknowns as linear functions of the remaining n-r unknowns, which are independ-

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1. Dickson, Theory of equations, 116-121.

ent variables, and the expressions for these r unknowns satisfy also the remaining n-r equations.

The theorem for homogeneous linear equations is similar except the determinants K are all zero. A particular case of the theorem is:

> A necessary and sufficient condition that a linear homogeneous equations in n unknowns shall have a set of solutions other than the trivial one in which each unknown is zero, is that the determinant of the coefficients be zero.

In order to study a system of m linear equations in n unknowns it will be convenient to introduce matrices. A matrix is a rectangular array of which the square array is a special case; a matrix is not one quantity but mn quantities; thus it does not have a definite value. When the column composed of the known terms $k_i$ is annexed to the matrix of the coefficients of the unknowns, one obtains the so-called augmented matrix.

If, of the determinants which can be formed from a given matrix, not all those of order r are zero, whereas all those of order greater than r are zero, the matrix is said to be of rank r. A very important theorem may now be stated:

> A system of m linear equations in n unknowns is consistent if, and only if, the rank of the matrix of the coefficients of the unknowns is equal to the rank of the augmented matrix. If the rank of the matrices is r, certain r of the equations determine uniquely r of the unknowns as linear functions of the remaining n-r unknowns, which are independent variables, and the expressions for these r unknowns satisfy also the remaining m-r equations.

Determinants are very useful in the study of the theory of linear dependence. Linear dependence is a generalization of the conception of proportionality.

The two sets of constants[1]

$$x'_1, x'_2, \ldots x'_n$$

$$x''_1, x''_2, \ldots x''_n$$

are said to be proportional to each other if two constants $c_1$ and $c_2$, not both zero, exist such that

$$c_1 x'_i + c_2 x''_i = 0 \qquad (i = 1, 2, \ldots n)$$

The m sets of n constants each,

$$x^i_1, x^i_2, \ldots x^i_n \qquad (i = 1, 2, \ldots m),$$

are said to be linearly dependent if m constants $c_1$, $c_2$, $\ldots c_m$, not all zero, exist such that

$$c_1 x'_i + c_2 x''_i + \ldots + c_m x^m_i = 0 \qquad (i = 1, 2, \ldots n).$$

If this is not the case, the sets of quantities are said to be linearly independent.

A necessary and sufficient condition for the linear dependence of m sets of n constants each, when $m \leqq n$, is that all the m-rowed determinants of the matrix

$$\begin{Vmatrix} x'_1 & x'_2 & \cdot & \cdot & \cdot & x'_n \\ x''_1 & x''_2 & \cdot & \cdot & \cdot & x''_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x^m_1 & x^m_2 & \cdot & \cdot & \cdot & x^m_n \end{Vmatrix}$$

should vanish.

Determinants are extensively used in practically all branches of higher algebra[2]. Determinants may be used to solve

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1. Bôcher, Higher algebra, 34-36.

2. Bôcher, Higher algebra, 315 p.

systems of equations which are not linear. For example, to eliminate the unknowns from the equations

$$ax^2 + bxy + cy^2 = 0$$

$$dx + ey = 0$$

multiply each term of the second equation, first by y and then by x, there becomes available three equations involving the three unknowns $x^2$, $xy$, $y^2$ as follows:

$$ax^2 + bxy + cy^2 = 0,$$

$$dxy + ey^2 = 0,$$

$$dx^2 + exy = 0.$$

If this system is consistent, the eliminant must be equal to zero, that is

$$\begin{vmatrix} a & b & c \\ 0 & d & e \\ d & e & 0 \end{vmatrix} = 0.$$

If the given equations are inconsistent, this determinant does not vanish. This process, due to Sylvester, may readily be generalized. It is known as the dialytic method of elimination.

Thus far all the applications have been in the field of algebra. When one considers the geometric interpretation of these algebraic properties, he will see that the theory of determinants is equally important in geometry as in algebra.

If the rectangular[1] coordinates of the vertices of a tri-

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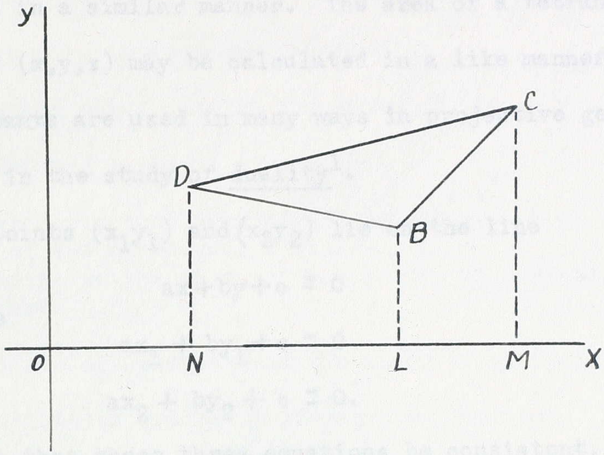
1. Scott and Mathews, Theory of determinants, 223-256.

angle are given, the area may be computed by determinants. Let the vertices of BCD be $(x_1y_1)$, $(x_2y_2)$, and $(x_3y_3)$. Then from the figure




it is seen that the area A of the triangle is given by

$$A = \text{trap. } MCDN - \text{trap. } MCBL - \text{trap. } LBDN$$

$$= \tfrac{1}{2}(y_2 + y_3)(x_2 - x_3) - \tfrac{1}{2}(y_2 + y_1)(x_2 - x_1) - \tfrac{1}{2}(y_3 + y_1)(x_1 - x_3);$$

or

$$2A = y_3x_2 - y_2x_3 + x_3y_1 - x_1y_3 + x_1y_2 - x_2y_1,$$

$$= \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

If the axes were oblique this determinant would have to be multiplied by the sine of the angle between the axes. Thus

$$2A = \sin(XOY)\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

One will get the negative of A, if he takes the vertices in the opposite order. It will be seen that the area of other polygons may be determined in a similar manner. The area of a tetrahedron given in terms of (x,y,z) may be calculated in a like manner.

Determinants are used in many ways in projective geometry. One may use them in the study of duality[1].

If the points $(x_1y_1)$ and $(x_2y_2)$ lie on the line

$$ax+by+c=0$$

one must have

$$ax_1+by_1+c=0$$

$$ax_2+by_2+c=0.$$

The condition that these three equations be consistent, viz.,

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0,$$

is the equation of the line joining the two points.

If the lines $(u_1v_1)$ and $(u_2v_2)$ pass through the point

$$au+bv+c=0$$

one must have

$$au_1+bv_1+c=0$$

$$au_2+bv_2+c=0.$$

The condition that these three equations be consistent, viz.,

$$\begin{vmatrix} u & v & 1 \\ u_1 & v_1 & 1 \\ u_2 & v_2 & 1 \end{vmatrix} = 0,$$

is the equation of the point of intersection of the two lines.

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1. Winger, Projective geometry, 17-18.

Determinants are useful also in the study of the characteristics of collineations and involutions. They are especially adapted to the study of the analytic treatment of conics. Determinants are used in many other ways in advanced geometry.[1]

Hessians are often used in the study of higher plane curves[2]. Hessians may be defined in terms of conics:

> The Hessian is the locus of all singularities of the first polars of the nth degree curve.
> The Hessian is the locus of the points of contact of the first polars.

Determinants often appear in the study of group theory[3], especially in sections on groups of linear substitutions.

Determinants are used in solving problems in civil engineering, such as the complex calculations that arise in the design of skyscrapers. The geodetic surveyer also has occasion to use determinants.

In the study of electricity, determinants are useful in solving a problem of the following type:- Given a Wheatstone bridge circuit. Apply Kirchhoff's first and second laws to derive an expression for the current in a generator. By using Kirchhoff's first law one gets four equations involving the current in the six branches of the system; and three more equations are obtained by using the second rule, these equations involve the same current and the voltage

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1. Graustein, Higher geometry.

2. Salmon, Higher plane curves.

3. Mathewson, Elementary finite groups.

also occurs in one of them. These equations can be solved by determinants to give an expression for $I_g$ in terms of the voltage.

In recent years statisticians are finding many new ways to use determinants to shorten their work. Some textbooks[1] in statistics are so largely based on determinants that several pages in the introductory chapter are devoted to the properties of determinants. Problems of multiple correlation, which are very lengthy, when solved by the old method, become relatively short when solved by a new method involving the use of determinants.

Another field in which determinants are useful is economics. The fact that a set of equations[2] will be independent and the solution will be determined when their Jacobian is not equal to zero is used in solving sets of interest equations.

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1. Thurstone, Vectors of mind.
2. Evans, Math. intro. to economics, 92.

# CONCLUSION

In this monograph it has been shown that historians have not been able to decide definitely who first thought of determinants. However, the Chinese and the Japanese used determinants many years before they were known to the western civilization. The theory of determinants began to spread in Europe about 1750; it spread very slowly for many years. In the last 120 years the developments have been largely in the direction of special forms.

Determinants originated in an attempt to solve simultaneous linear equations. The theory of determinants is very closely related to the theory of permutations, for the reason that the type of permutations of each term of a determinant fixes its algebraic sign. There are many ways in which determinants may be transformed; such as interchanging rows and columns, interchanging rows or columns, adding the elements of one row to the corresponding elements of another row, etc. There are many ways of expanding determinants; each method has certain advantages and disadvantages.

The application of determinants has led to the development of many new special forms. The use and application of determinants now extends into many sciences, particularly in the physical sciences, the biological sciences, and the social sciences.

The uses of higher order determinants have become very extensive, but fortunately, since the expansion of determinants is symmetrical, it has been possible to develop a machine to evaluate these determinants. This new machine[1], which is known as the simultaneous linear equator, is capable of solving nine simultaneous linear equations involving nine unknowns. It was designed and built by Dr. John B. Wilbur of the department of Civil Engineering of Massachusetts Institute of Technology.

The machine was designed originally for the solution of problems in civil engineering, but this calculator will, in all probability, soon be used in many other fields of engineering and research. Once the elements are set up on the calculator, a single movement of the mechanism gives the value of the determinant in a few seconds.

The calculator contains more than 13,000 separate parts, weighs a ton, has 600 feet of flexible steel tape and nearly 1000 ball-bearing pulleys.

From present indications one may expect many new applications of determinants in practically all fields of science in the next few years. Research workers of various fields will find more ways in which to use determinants, as they become better acquainted with the theory of determinants.

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1. Wilbur, Jour. of engin. educ., XXVII, no. 7, Mar. 1937. 524.

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